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On Polynomials with Largest Coefficient Sums

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Let F_j denote the linear functional that assigns to a real polynomial $P_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ its jth partial coefficient sum $F_j(P_n) = a_0 + a_1 + a_2 + \cdots + a_j$ $(1 \leqslant j \leqslant n-1; n \geqslant 5)$. It is demonstrated that a polynomial P_n^* which is extremal for F_j (i.e., $\|P_n^*\|_\infty = 1$ (uniform norm on $I = [-1, 1]$) and $F_j(P_n^*) = \|F_j\|$) must have d alternation points on I, where $n-3 \leqslant d \leqslant n+1$. This result complements the author's previous one [*Math. Z.* **182** (1983), 549–552] stating that about "half" the time, namely, if $j \equiv n \pmod 2$, the nth Chebyshev polynomial of the first kind, $\pm T_n$, which posseses $d = n+1$ alternation points on I, is extremal for F_j. Known results on this subject are surveyed and additional topics such as uniqueness of polynomials with largest coefficient sums and practical estimation of $|F_j(P_n)|$ are included, to make the paper self-contained.

1. INTRODUCTION AND SURVEY OF RESULTS

Let $\mathbb{P}_n$ denote the linear space of real polynomials P_n of degree not exceeding $n \in \mathbb{N}$, normed in Chebyshev's sense, i.e.,

$$\|P_n\|_\infty = \max_{x \in I} |P_n(x)|, \tag{1}$$

where $I = [-1, 1]$, and let

$$B_n = \{P_n \in \mathbb{P}_n : \|P_n\|_\infty \leqslant 1\} \tag{2}$$

denote the unit ball in $\mathbb{P}_n$.

A prominent member of B_n is the nth Chebyshev polynomial of the first kind, $T_n = \sum_{k=0}^{n} t_k^{(n)} \operatorname{id}^k$, which is, simultaneously with n, an even or odd polynomial; cf. [12]. By id we denote the identical function given by $\operatorname{id}(x) = x$.

About a hundred years ago, V. Markov obtained sharp estimates for

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each single coefficient of an arbitrary polynomial $P_n = \sum_{k=0}^{n} a_k \,\mathrm{id}^k \in B_n$ in terms of the coefficients of T_n and $T_{n-1} \in B_n$,

$$|a_j| \leqslant \begin{cases} |t_j^{(n)}|, & \text{if } j \equiv n \pmod 2 \\ |t_j^{(n-1)}|, & \text{if } j \equiv n-1 \pmod 2 \end{cases} \tag{3}$$

(cf., e.g., [5, p. 56]). The integer numbers $t_j^{(n)}$ are explicitly known. In this paper we are concerned with the problem of determining polynomials P_n from B_n which have largest partial sums of coefficients. Thus we are interested in the structure of those polynomials which give the norm of the linear coefficient functionals $F_j\colon \mathbb{P}_n \to \mathbb{R}$, where

$$F_j(P_n) = a_0 + a_1 + a_2 + \cdots + a_j \qquad \left(0 \leqslant j \leqslant n;\, P_n = \sum_{k=0}^{n} a_k \,\mathrm{id}^k\right). \tag{4}$$

We shall refer to a $P_n^* \in B_n$ as extremal for F_j if $\|P_n^*\|_\infty = 1$ and

$$F_j(P_n^*) = \|F_j\| = \sup_{P_n \in B_n} |F_j(P_n)|.$$

We note in passing that the trivial upper bound $|F_j(P_n)| \leqslant \sum_{k=0}^{j} |a_k|$, which can be evaluated via Markov's inequalities (3) yields unreasonable results; see also Theorem 5 below.

A first step towards the posed problem was made by Reimer and Zeller [11] during an investigation into the numerical stability of evaluation schemes for polynomials. They showed in [11, Satz 1] that the partial coefficient sums of the even resp. odd component (depending on $n \in \mathbb{N}$) of $P_n \in B_n$ are maximized in absolute value by those of $\pm T_n$.

The same conclusion was established by Rivlin [12, p. 94] with a different method of proof under the weaker assumption $P_n \in C_n$. Here,

$$C_n = \{P_n \in \mathbb{P}_n\colon |P_n(\hat{x}_{k,n})| \leqslant 1 \text{ for } k = 0, 1, 2, ..., n\} \tag{5}$$

and $\hat{x}_{k,n}$ denotes the points where T_n attains its extreme values ± 1 on I,

$$\hat{x}_{k,n} = \cos((n-k)\,\pi/n), \qquad k = 0, 1, 2, ..., n. \tag{6}$$

Note that B_n is a proper subset of C_n $(n \geqslant 2)$; note also that the lower index of summation in [12, Formula (2.38)] should read $j = i$.

In [8] we gave an alternative proof for this result in a more general setting, and added to it the sharp upper bounds for the partial coefficient sums of the even resp. odd component of $P_n \in C_n$ if n is odd resp. even. In summary we thus have (cf. [8]):

THEOREM 1. *Let* $P_n = \sum_{k=0}^{n} a_k \,\mathrm{id}^k \in C_n$. *Then*

$$\text{(i)}\quad |a_0 + a_2 + a_4 + \cdots + a_j| \leqslant |t_0^{(n)} + t_2^{(n)} + t_4^{(n)} + \cdots + t_j^{(n)}|,\quad \text{if } n \text{ is even and } j \equiv n \pmod 2; \tag{7}$$

$$\text{(ii)}\quad |a_1 + a_3 + a_5 + \cdots + a_j| \leqslant |t_1^{(n)} + t_3^{(n)} + t_5^{(n)} + \cdots + t_j^{(n)}|,\quad \text{if } n \text{ is odd and } j \equiv n \pmod 2; \tag{8}$$

$$\text{(iii)}\quad |a_0 + a_2 + a_4 + \cdots + a_j| \leqslant |c_0 + c_2 + c_4 + \cdots + c_j|,\quad \text{if } n \text{ is odd and } j \equiv n-1 \pmod 2; \tag{9}$$

$$\text{(iv)}\quad |a_1 + a_3 + a_5 + \cdots + a_j| \leqslant |c_1 + c_3 + c_5 + \cdots + c_j|,\quad \text{if } n \text{ is even and } j \equiv n-1 \pmod 2. \tag{10}$$

The coefficients c_k *stem from the Rogosinski polynomial* $\prod = \prod_{n-1} = \sum_{k=0}^{n-1} c_k \,\mathrm{id}^k \in C_n$; cf. [13].

Let us now turn to the original problem of determining the elements of B_n which maximize the unrestricted partial coefficient sums $F_j(P_n)$. Our main result (Theorem 3 below) gives insight into the structure of these extremal polynomials. They are determined by their number d of alternation points on I, and among them are, in particular, the Chebyshev polynomials ($d = n+1$), the Zolotarev polynomials ($d = n$), and the Achieser polynomials ($d = n-1$). Detailed information on these classical polynomial families can be found in [1, 3, 4, 12] or [14]. The proof of Theorem 3 is based on the well-known characterization theorem of best Chebyshev approximations; cf. [12, Theorem 2.5]. The question of uniqueness of polynomials with largest coefficient sums is discussed in some detail in Section 3 below. In the final section of this paper we determine the extremal polynomials for F_j if $1 \leqslant n \leqslant 4$ and provide a practical estimate for $|F_j(P_n)|$ if $n \geqslant 5$ in terms of T_n. It is in fact a lucky coincidence that in at least "half" the time, namely if $j \equiv n \pmod 2$, an extremal polynomial for F_j is given by the Chebyshev polynomial $\pm T_n$. We have shown this result in [9]:

THEOREM 2. *Let* $P_n = \sum_{k=0}^{n} a_k \,\mathrm{id}^k$ *and let* $P_n \in B_n$ *or* $P_n \in C_n$. *Then*

$$|F_j(P_n)| \leqslant |F_j(\pm T_n)|, \qquad \text{if} \quad j \equiv n \pmod 2, \tag{11}$$

and hence in particular

$$\|F_j\| = |F_j(\pm T_n)|, \qquad \text{if} \quad j \equiv n \pmod 2. \tag{12}$$

2. THE MAIN RESULT

$P_n \in B_n$ is said to alternate on I d times ($d \geqslant 2$) if there exist points $z_1 < z_2 < \cdots < z_d$ from I (alternation points) with the property $|P_n(z_w)| = 1$

for $w=1, 2,..., d$ and $P_n(z_w)\, P_n(z_{w+1}) = -1$ for $w = 1, 2, ..., d-1$. In what follows it suffices to assume $n \geqslant 5$ (cf. Proposition 5 below). Since $\|F_0\| = 1$ (compare inequality (3)) and $\|F_n\| = 1$ in virtue of $|F_n(P_n)| = |P_n(1)|$, it is enough to consider only those functionals F_j with index $j \in \{1, 2, ..., n-1\}$.

THEOREM 3. *Let $P_n^* \in B_n$, $n \geqslant 5$, be extremal for the functional F_j, $1 \leqslant j \leqslant n-1$. Then P_n^* alternates on I d times, where $n-3 \leqslant d \leqslant n+1$.*

Proof. Let $P_n^* = \sum_{k=0}^n a_k^* \,\mathrm{id}^k$ be extremal for F_j, $1 \leqslant j \leqslant n-1$. Put

$$P = \sum_{k=0}^{j} a_k^* \,\mathrm{id}^k \qquad \text{and} \qquad P_n^\# = P - P_n^*. \tag{13}$$

This implies $F_j(P_n^\#) = 0$ so that

$$P_n^\# \in K = \mathrm{kernel}(F_j) = \mathrm{span}(1 - \mathrm{id}^j, \mathrm{id} - \mathrm{id}^j, ..., \mathrm{id}^{j-1} - \mathrm{id}^j, \mathrm{id}^{j+1}, ..., \mathrm{id}^n). \tag{14}$$

We claim that $P_n^\#$ is a best approximation to $P \in \mathbb{P}_n \backslash K$ from K, i.e.,

$$\|P - P_n^\#\|_\infty \leqslant \|P - P_n^0\|_\infty \qquad \text{for all} \quad P_n^0 \in K:$$

On the one hand we obtain (with $P_n^0 \in K$)

$$\|F_j\| = |F_j(P)| = |F_j(P) - F_j(P_n^0)| = |F_j(P - P_n^0)| \leqslant \|F_j\|\ \|P - P_n^0\|_\infty,$$

i.e., $1 \leqslant \|P - P_n^0\|_\infty$ for all $P_n^0 \in K$.

On the other hand we get $1 = \|P_n^*\|_\infty = \|P - P_n^\#\|_\infty$.

According to [12, Theorem 2.5] there exist distinct points $x_1 < x_2 < \cdots < x_r$ from the set

$$E(P_n^*; I) = \{x \in I: |P_n^*(x)| = 1\} \tag{15}$$

of critical points of P_n^*, and positive numbers $\mu_1, \mu_2, ..., \mu_r$ such that

$$\sum_{i=1}^{r} \mu_i P_n^*(x_i)\, P_n^0(x_i) = 0 \qquad \text{for all} \quad P_n^0 \in K, \tag{16}$$

where $1 \leqslant r \leqslant \dim(K) + 1 = n+1$.

Suppose that $r \leqslant n-2$. We proceed to construct some $Q_n \in K$ which violates Eq. (16) so that $n-1 \leqslant r \leqslant n+1$ holds true. Set

$$R(x) = \prod_{l=1}^{r} (x - x_l) = \sum_{k=0}^{r} b_k x^k \tag{17}$$

and

$$Q_n(x) = P_n^*(x) + (\alpha + \beta x + \gamma x^2)\, R(x) \tag{18}$$

with unspecified real scalars α, β, γ. Obviously, $Q_n \in \mathbb{P}_n$ and $Q_n(x_i) = P_n^*(x_i)$ for $i = 1, 2, ..., r$ so that

$$\sum_{i=1}^{r} \mu_i P_n^*(x_i)\, Q_n(x_i) > 0. \tag{19}$$

Furthermore,

$$\begin{aligned} F_j(Q_n) &= F_j(P_n^*) + \alpha F_j(R) + \beta F_j(\mathrm{id}\, R) + \gamma F_j(\mathrm{id}^2\, R) \\ &= F_j(P_n^*) + \alpha F_j(R) + \beta F_{j-1}(R) + \gamma F_{j-2}(R), \end{aligned} \tag{20}$$

with $F_{-1}(R) = 0$ and $F_j(P_n^*) > 0$.

(I) Let $r = n-2$. We distinguish several cases concerning the index j to show that the scalars α, β, γ can be chosen in such a manner that $F_j(Q_n) = 0$, i.e., $Q_n \in K$.

Case 1. $j = 1$. Then, $F_1(Q_n) = F_1(P_n^*) + \alpha F_1(R) + \beta F_0(R)$. It is impossible that $F_1(R) = F_0(R) = 0$, since then R would have a double root at $x = 0$. Thus take either $\alpha = 0$ and $\beta = -F_1(P_n^*)/F_0(R)$ or $\alpha = -F_1(P_n^*)/F_1(R)$ and $\beta = 0$ to get $F_1(Q_n) = 0$.

Case 2. $j \in \{2, 3, ..., n-2\}$. It is then impossible that $F_j(R) = F_{j-1}(R) = F_{j-2}(R) = 0$, since in this case the two consecutive coefficients b_j and b_{j-1} of R would vanish although R has only real roots, a contradiction to Descartes' rule of signs; cf. [7, Satz 13.4]. Hence at least one of the numbers $F_j(R)$, $F_{j-1}(R)$, or $F_{j-2}(R)$ is different from zero and it is then obvious that in Eq. (20) α, β, and γ can be appropriately chosen so as to yield $F_j(Q_n) = 0$.

Case 3. $j = n-1$. Then, $F_{n-1}(Q_n) = F_{n-1}(P_n^*) + \alpha R(1) + \beta R(1) + \gamma F_{n-3}(R)$. If $R(1) \neq 0$, the choice $\beta = \gamma = 0$ and $\alpha = -F_{n-1}(P_n^*)/R(1)$ gives $F_{n-1}(Q_n) = 0$; if $R(1) = 0$, then $F_{n-3}(R) \neq 0$ since otherwise the leading coefficient b_{n-2} of R (which is in fact 1) would vanish. Thus we may take $\alpha = \beta = 0$ and $\gamma = -F_{n-1}(P_n^*)/F_{n-3}(R)$ to force $F_{n-1}(Q_n) = 0$.

(II) Let $1 \leqslant r \leqslant n-3$. Choosing distinct points $y_1, y_2, ..., y_{n-r-2} \in I \backslash \{x_1, ..., x_r\}$ and replacing $R(x)$ in Eq. (17) by

$$\tilde{R}(x) = \prod_{l=1}^{r} (x - x_l) \prod_{u=1}^{n-r-2} (x - y_u) \tag{21}$$

we can proceed as before and eventually get $Q_n \in K$ if $1 \leqslant r \leqslant n-2$. Thus there are r critical points of P_n^* on I, where $n-1 \leqslant r \leqslant n+1$.

Remarks. (i) It follows from the above considerations that in the special cases $j=1$ and $j=n-1$ a contradiction to Eq. (16) can be produced by assuming $r=n-1$ in place of $r=n-2$ and putting $\gamma=0$ in Eq. (18). Hence in these two marginal cases one has actually $r \in \{n, n+1\}$ critical points of P_n^* on I.

(ii) It is tempting to make the same assumption $r=n-1$ and $\gamma=0$ in the cases $j \in \{2, 3, ..., n-2\}$ as well. But the attempt fails since for these values of j there exist polynomials R with only real (simple) roots such that

$$F_j(R) = F_{j-1}(R) = 0. \tag{22}$$

Knowing that P_n^* has $r \in \{n-1, n, n+1\}$ critical points on I we are now in a position to bound the number d of alternation points of P_n^* on I:

(A) $r=n+1$. By [12, Theorem 2.12] we have either $P_n^* = \pm T_n$ or $P_n^* = \pm 1$ (constant). But $|F_j(\pm 1)| = 1$ is surely smaller than, for example, $|F_j(\pm T_n)|$ or $|F_j(\pm T_{n-1})|$ so that ± 1 cannot be extremal. Thus $P_n^* = \pm T_n$ which means that $d=n+1$ and the alternation points being the $\hat{x}_{k,n}$ from Eq. (6).

(B) $r=n$. The first derivative $P_n^{*\prime}$ of P_n^* vanishes $n-2$ times at the interior points $x_2 < \cdots < x_{n-1}$ since $P_n^* \in B_n$. If there were, induced by these points, two or more subintervals $[x_q, x_{q+1}]$ with the property $P_n^*(x_q)\, P_n^*(x_{q+1}) = 1$, then, by Rolle's Theorem, $P_n^{*\prime}$ would have at least n roots in I and so $P_n^{*\prime} = 0$ and $P_n^* = 1$, a contradiction.

Therefore there exists at most one subinterval $[x_q, x_{q+1}]$ induced by the interior points where P_n^* does not alternate. Suppose first that P_n^* does not alternate on the interior interval $[x_q, x_{q+1}]$ so that $P_n^{*\prime}$ has $n-1$ roots in I. If we than had, in addition, $P_n^*(x_1)\, P_n^*(x_2) = 1$ or $P_n^*(x_{n-1})\, P_n^*(x_n) = 1$, this would imply that $P_n^{*\prime}$ posseses more than $n-1$ roots in I, an impossibility. Hence P_n^* must alternate on the boundary intervals $[x_1, x_2]$ and $[x_{n-1}, x_n]$ (with $P_n^{*\prime}(x_1) \neq 0 \neq P_n^{*\prime}(x_n)$) giving a total of $d=n-1$ alternation points.

Suppose next that P_n^* alternates at all interior points. With regard to the zeros of $P_n^{*\prime}$ we conclude that either P_n^* alternates on $[x_1, x_2]$ and on $[x_{n-1}, x_n]$ (with $P_n{}^{*\prime}(x_1) \neq 0$ or $P_n^{*\prime}(x_n) \neq 0$) or P_n^* does not alternate on exactly one of these boundary intervals (with $P_n^{*\prime}(x_1) \neq 0$ resp. $P_n^{*\prime}(x_n) \neq 0$). This leads to $d=n$ or $d=n-1$ alternation points of P_n^* on I.

(C) $r=n-1$. By similar considerations as in (B) we get $d=n-1$ or $d=n-2$ or even $d=n-3$ alternation points of P_n^* on I. The latter case occurs, for example, if P_n^* alternates at the interior points $x_2 < \cdots < x_{n-2}$ but neither on $[x_1, x_2]$ nor on $[x_{n-2}, x_{n-1}]$ (with $P_n^{*\prime}(x_1) \neq 0 \neq P_n^{*\prime}(x_{n-1})$). ∎

3. Uniqueness of Polynomials with Largest Coefficient Sums

We know from Theorem 2 that $\sup_{P_n \in B_n} |F_j(P_n)| = \sup_{P_n \in C_n} |F_j(P_n)| = |F_j(\pm T_n)|$, provided that $j \equiv n \pmod 2$. Examining the proof in [9] we observe that the value of P_n at $x = \hat{x}_{0,n} = -1$ does not matter. Therefore we can state a slightly more general version of Korollar 1 in [9] which includes polynomials P_n satisfying $|P_n(-1)| > 1$. For convenience, we first introduce the set

$$D_n = \{P_n \in \mathbb{P}_n : |P_n(\hat{x}_{k,n})| \leqslant 1 \text{ for } k = 1, 2, ..., n\} \tag{23}$$

and note that $B_n \subset C_n \subset D_n$.

Proposition 1.

$$\sup_{P_n \in D_n} |F_j(P_n)| = |F_j(\pm T_n)|, \qquad \text{if} \quad j \equiv n \pmod 2. \tag{24}$$

We now consider the question of uniqueness of polynomials with largest partial coefficient sums in D_n, C_n, and B_n. From the proof in [9] also follows that besides $\pm T_n$ there are infinitely many elements in D_n (and in C_n) that attain the sup in Eq. (24).

Proposition 2. *Every polynomial $\hat{P}_n \in D_n$ satisfying either $\hat{P}_n(\hat{x}_{k,n}) = (-1)^{n-k}$ or $\hat{P}_n(\hat{x}_{k,n}) = -(-1)^{n-k}$ for $k = 1, 2, ..., n$ yields*

$$\sup_{P_n \in D_n} |F_j(P_n)| = |F_j(\hat{P}_n)|, \qquad \text{if} \quad j \equiv n \pmod 2. \tag{25}$$

Example 1. Consider the parameterized polynomial $\hat{P}_{5,t} \in D_5$ (with parameter $t \in \mathbb{R}$) given by

$$\hat{P}_{5,t}(x) = 0.1(1+t) + (4.9 - 0.1t)\,x - 1.2(1+t)\,x^2 + (1.2t - 18.8)\,x^3 + 1.6(1+t)\,x^4 + 1.6(9-t)\,x^5.$$

It satisfies $\hat{P}_{5,t}(-1) = t$, $\hat{P}_{5,t}(\hat{x}_{k,5}) = (-1)^{5-k}$ for $k = 1, 2, 3, 4, 5$, and gives the same maximizing partial coefficient sums $|F_j(\hat{P}_{5,t})|, j \in \{1, 3, 5\}$, as $\pm T_5$, where $T_5(x) = 5x - 20x^3 + 16x^5$ (i.e., $T_5 = \hat{P}_{5,-1}$). If $|t| \leqslant 1$, $\hat{P}_{5,t}$ belongs to C_n.

However, if we require a maximizing polynomial $\hat{P}_n \in D_n$ (or $\hat{P}_n \in C_n$) (cf. Proposition 2) to be an element of B_n we obtain $\hat{P}_n = \pm T_n$. This follows from our next statement.

PROPOSITION 3. *Let* $P_n \in B_n, n \geqslant 2$, *satisfy* $P_n(\hat{x}_{k,n}) = (-1)^{n-k}$ *for* $k = 1, 2, ..., n$. *Then*, $P_n = T_n$.

Proof. Consider the polynomial $G_n = P_n - T_n$. As G_n vanishes at $\hat{x}_{k,n}$ $(k = 1, 2, ..., n)$ we get

$$G_n(x) = K_1 \prod_{k=1}^{n} (x - \hat{x}_{k,n}) \qquad (K_1 \text{ a constant})$$

and

$$G_n'(x) = K_1 \sum_{m=1}^{n} \prod_{\substack{k=1 \\ k \neq m}}^{n} (x - \hat{x}_{k,n}). \tag{26}$$

Since P_n, $T_n \in B_n$ we deduce that G_n' vanishes at the interior points $\hat{x}_{k,n}$ of I $(k = 1, 2, ..., n-1)$ and thus

$$G_n'(x) = K_2 \prod_{k=1}^{n-1} (x - \hat{x}_{k,n}) \qquad (K_2 \text{ a constant}). \tag{27}$$

Equating (26) and (27) we deduce at $x = \hat{x}_{n,n} = 1$ the identity $K_1 = K_2$, and a comparison of the coefficients of x^{n-1} gives $K_2 = nK_1$. This implies $K_1 = K_2 = 0$ and $P_n = T_n$. ∎

Next, we turn to the question of the uniqueness of extremal polynomials in B_n.

THEOREM 4. *The functional* F_j $(1 \leqslant j \leqslant n-1; n \geqslant 5)$ *has a unique extremal element in* B_n.

Proof. Reconsider Eq. (16) above and choose for $P_n^0 \in K$ the special polynomial $\bar{P}_n^0 = F_j(P_n) P_n^* - \|F_j\| P_n$, where $P_n \in \mathbb{P}_n$ is arbitrary. Obviously, $\bar{P}_n^0 \in K$, and, rearranging Eq. (16), we deduce (see also the proof of Theorem 2.13 in [12]):

$$\sum_{i=1}^{r} \mu_i P_n^*(x_i) \bar{P}_n^0(x_i) = 0,$$

$$F_j(P_n) \sum_{i=1}^{r} \mu_i (P_n^*(x_i))^2 = \sum_{i=1}^{r} \mu_i P_n^*(x_i) \|F_j\| P_n(x_i),$$

$$F_j(P_n) = \sum_{i=1}^{r} \mu_i P_n^*(x_i) \|F_j\| \mu^{-1} P_n(x_i),$$

$$F_j(P_n) = \sum_{i=1}^{r} \alpha_i P_n(x_i), \tag{28}$$

where $\mu = \mu_1 + \mu_2 + \cdots + \mu_r > 0$ and $\alpha_i = \mu_i P_n^*(x_i) \|F_j\| \mu^{-1}$. We know from the proof of Theorem 3 that in Eq. (28), which is a canonical representation of F_j in the sense of [12, p. 84], there are $r \in \{n-1, n, n+1\}$ terms. Even if we assume the "worst case" concerning the choice of r, $r = n-1$, and the distribution of points $x_i \in I$, $-1 = x_1 < x_2 < \cdots < x_{n-2} < x_{n-1} = 1$, we obtain uniqueness of the extremal element for F_j since the sufficient condition for the uniqueness as given in [12, Theorem 2.17] is now in force; this condition reads here, with the notation $E(x) = 1$, if $x = 1$ or $x = -1$, and $E(x) = 2$, if $-1 < x < 1$:

$$\sum_{i=1}^{r} E(x_i) = \sum_{i=1}^{n-1} E(x_i) = 2n - 4 > n, \qquad \text{provided that} \quad n \geqslant 5. \quad \blacksquare$$

The following statement reveals in conjunction with Theorem 3 above that extremal elements for F_j which have $n-3 \leqslant d \leqslant n$ alternation points on I can occur only if $j \equiv n-1 \pmod 2$.

PROPOSITION 4. *The unique extremal element in B_n for F_j $(n \geqslant 5)$ is*

$$T_n, \qquad \textit{if} \quad j = n - 4h \ (h = 1, 2, 3, \ldots) \tag{29}$$

$$-T_n, \qquad \textit{if} \quad j = n - 2 - 4h \ (h = 0, 1, 2, \ldots). \tag{30}$$

Proof. According to Theorem 2, an extremal element for F_j $(1 \leqslant j \leqslant n-1; j \equiv n \pmod 2)$ is given by T_n or by $-T_n$. In either case it is uniquely determined by Theorem 4 if $n \geqslant 5$. An analysis of the alternating signs of $S_j = t_j^{(n)} + t_{j-2}^{(n)} + t_{j-4}^{(n)} + \cdots$ (cf. Exercise 1.2.19 in [12]) finally establishes (29) and (30). $\blacksquare$

4. PRACTICAL CALCULATION OF PARTIAL COEFFICIENT SUMS

For small values of n $(1 \leqslant n \leqslant 4)$ extremal elements for F_j can be calculated explicitly; but the calculation is rather lenthy so that we state the results without proof.

PROPOSITION 5. *If $n \in \{1, 2, 3, 4\}$ then either $\pm T_n$ or $\pm T_{n-1}$ are extremal for F_j $(0 \leqslant j \leqslant n)$, except in the following three instances*:

(a) $n = 2$ *and* $j = 1$. *An extremal polynomial is given by* $P_2^*(x) = (7 + 8x - 8x^2)/9$ *and one gets* $F_1(P_2^*) = \|F_1\| = 15/9 = 1.66666\ldots$ (31)

(b) $n = 3$ *and* $j = 2$. *An extremal polynomial is given by the (improper) Zolotarev polynomial* $P_3^*(x) = -T_3(((8 + \sqrt{22})x - 6 + \sqrt{22})/14) =$

$-(270 - 17\sqrt{22})/686 + ((576 + 261\sqrt{22})/686)\,x + ((246 + 15\sqrt{22})/343)\,x^2 - ((520 + 107\sqrt{22})/343)\,x^3$ *and one gets*

$$F_2(P_3^*) = \|F_2\| = (57 + 22\sqrt{22})/49 = 3.26916.... \tag{32}$$

(c) $n = 4$ *and* $j = 1$. *An extremal polynomial is given by the (proper) Zolotarev polynomial with parameter* t, $1 < t < 1 + \sqrt{2}$,

$$Z_{4,t}(x) = a_0(t) + a_1(t)\,x + a_2(t)\,x^2 + a_3(t)\,x^3 + a_4(t)\,x^4,$$

where

$$\begin{aligned}
a_0(t) &= K_{a,b}(-a^5 + a^4(-2 + 3b) + a^3(-1 + 6b - 3b^2) \\
&\qquad + a^2(3b + 2b^2 + b^3) + a(3b^2 - 2b^3) - b^3) \\
a_1(t) &= K_{a,b}(a^2(-16b + 8b^2) + a(-12b + 8b^2 - 4b^3)) \\
a_2(t) &= K_{a,b}(a^2(8 - 16b) + a(6 - 4b + 2b^2) + 6b - 4b^2 + 2b^3) \\
a_3(t) &= K_{a,b}(8a^2 + 8ab + 8b - 4b^2 - 4) \\
a_4(t) &= K_{a,b}(-6a + 2b - 4)
\end{aligned}$$

and

$$K_{a,b} = (b - a)^{-3}(1 + a)^{-2}$$

with

$$\begin{aligned}
a &= a(t) = (1 - 3t - t^2 - t^3)/(1 + t)^3, \\
b &= b(t) = (1 + t + 3t^2 - t^3)/(1 + t)^3.
\end{aligned}$$

One finds $\|Z_{4,t}\|_\infty = 1$ *and that* $Z_{4,t}$ *alternates on* I *at* $-1 < a < b < 1$:

$$Z_{4,t}(-1) = 1,\ Z_{4,t}(a) = -1,\ Z_{4,t}(b) = 1,\ Z_{4,t}(1) = -1.$$

The (parameterized) value $F_1(Z_{4,t}) = a_0(t) + a_1(t)$ *is largest, as differentiation with respect to* t *shows, if we choose for* t *the unique positive root of the algebraic equation of degree* 14,

$$\begin{aligned}
0 &= 3 + 16t + t^2 + 64t^3 + 47t^4 + 96t^5 + 181t^6 + 64t^7 \\
&\quad + 89t^8 + 16t^9 - 45t^{10} - 11t^{12} - 9t^{14},
\end{aligned}$$

i.e., $t = t^* = 1.52539...$, *which yields*

$$F_1(P_4^*) = F_1(Z_{4,t^*}) = \|F_1\| = 3.22652.... \quad \blacksquare \tag{33}$$

Given an arbitrary $n \in \mathbb{N}$ it is not possible to determine the coefficients of P_n^* in explicit power form so that numerical methods have to be applied. However, combining V. Markov's inequalities (3) with Theorem 2, we obtain a convenient bound for the partial coefficient sums of $P_n \in B_n$. This bound is sharp if $j \equiv n (\text{mod } 2)$ and improves Satz 2 in [11]:

THEOREM 5. *Let* $P_n = \sum_{k=0}^{n} a_k \, \mathrm{id}^k \in B_n$, *then*

$$|F_j(P_n)| \leqslant \begin{cases} |F_j(T_n)|, & \text{if } j \equiv n \ (\text{mod } 2) \\ |F_{j-1}(T_n)| + |t_j^{(n-1)}|, & \text{if } j \equiv n-1 \ (\text{mod } 2). \end{cases} \tag{34}$$

EXAMPLE 2. Let $n = 8$ and $P_8 = \sum_{k=0}^{8} a_k \, \mathrm{id}^k \in B_8$. Since

$$T_7(x) = -7x + 56x^3 - 112x^5 + 64x^7$$

and

$$T_8(x) = 1 - 32x^2 + 160x^4 - 256x^6 + 128x^8$$

we obtain the inequalities

$$|a_0| \leqslant 1, \ |a_0 + a_1| \leqslant 8, \ |a_0 + a_1 + a_2| \leqslant 31, \ |a_0 + \cdots + a_3| \leqslant 87,$$
$$|a_0 + \cdots + a_4| \leqslant 129, \ |a_0 + \cdots + a_5| \leqslant 241, \ |a_0 + \cdots + a_6| \leqslant 127,$$
$$|a_0 + \cdots + a_7| \leqslant 191, \ |a_0 + \cdots + a_8| \leqslant 1.$$

5. Closing Remarks

(i) The case $n = 4, j = 3$ in Proposition 5 reveals that a Chebyshev polynomial can be extremal for F_j even if $j \equiv n - 1$ (mod 2).

(ii) The statement and proof of Theorem 3 can be easily carried over to the coefficient functional H_j given by $H_j(P_n) = a_j + a_{j+1} + \cdots + a_n$, which describes the backward partial sums or Horner sums; cf. [10].

(iii) The problem of maximizing the coefficient sums of a complex polynomial is dealt with in [6].

(iv) Interesting results on the coefficient sums of the Rudin–Shapiro polynomials can be found in [12, p. 128; 2].

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